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Applications of geometrical criteria for transition to Hamiltonian chaos

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Using a recently developed geometrical method, we study the transition from order to chaos in an important class of Hamiltonian systems. We show agreement between this geometrical method and the surface of section technique applied to detect chaotic behavior. We give, as a particular illustration, detailed results for an important class of potentials obtained from the perturbation of an oscillator Hamiltonian by means of higher-order polynomials.

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I. INTRODUCTION

The problem of detecting the transition from order to chaos in Hamiltonian systems has become increasingly important in Hamiltonian chaos theory.

Many authors [1,2] have used geometrical methods to analyze the stability of a given Hamiltonian by constructing the manifold of solutions of the equations of motion; negative curvature corresponds to instability of the motion [3], and as seen by examining the deviation between nearby geodesic orbits that flow on the manifold, the motion is highly sensitive to initial conditions. On the other hand, positive curvature implies stability. The construction of an appropriate manifold (defined by its metric and connection form) is a crucial step to detect the stability of the motion.

One of the most important classes of Hamiltonians, as studied later on, has potentials obtained from the perturbation of an oscillator by higher-order polynomials (anharmonicity). Finding the transition from order to chaos in this class of systems has been highly elusive [1], and in some methods, chaotic behavior cannot be directly seen.

We use a method [4] recently developed to analyze this class of Hamiltonian systems in two dimensions in search of the transition point from order to chaos. The criteria depend on energy as well as parameters in the Hamiltonian. We show absolute agreement between our method and the well-known technique of surface of section (Poincaré plots).

II. THEORY

Casetti *et al.* [1] and others [2] have discussed the application of the Jacobi metric [5] [$g_{ij}=\delta_{ij}(E-V)$] to formulate criteria for the occurrence of instability in Hamiltonian systems. The Jacobi metric results in an invariant line interval which corresponds to a parameter s corresponding to the action rather than the time t. Transforming the variables in the resulting geodesic equations to t, one finds the standard Hamilton equations.

These authors have identified a class of Hamiltonians, those which at small distances become dominated by oscillatorlike terms, for which the application of this method is not a directly useful criterion in terms of a definite sign of curvature of the manifold. They nevertheless find a useful criterion in terms of the changes of curvature over space.

We have introduced an alternative formulation of the geometrical picture [4] which is directly effective for a large, rather unrestricted set of Hamiltonian systems, as well as for Hamiltonians which are dominated by oscillatorlike potentials at small distances. The method that we have developed results in an embedding of the Hamiltonian motion in a system of coordinates which are also subject to local coordinate covariance, and the geodesic deviation equations provide a very sensitive and effective criterion for stability.

In this system of general coordinates the Hamiltonian motion is described by the geodesic formula

$$\ddot{y}^{\ell} = -M^{\ell}_{mn}\dot{y}^m\dot{y}^n, \tag{1}$$

where

$$M^{\ell}_{mn} \equiv \frac{1}{2}g^{\ell k}\frac{\partial g_{nm}}{\partial y^k}. \tag{2}$$

Here, the metric tensor is related by general coordinate transformation to the conformal form

$$g_{ij} = \delta_{ij}\frac{E}{E-V}, \tag{3}$$

in the special coordinate system in which the Hamiltonian takes on the form

$$H = \frac{p^{i^2}}{2M} + V(y). \tag{4}$$

In these special coordinates the geodesic equation takes on the form

$$\ddot{y}^{\ell} = -\frac{\partial V}{\partial y^{\ell}}. \tag{5}$$

In general coordinates, however, the geodesic deviation computed from (1) provides us with a condition for stability which takes into account the curvature of the space of solutions; one may then substitute into the formula for geodesic deviation the special coordinate set and obtain a criterion which is stronger than direct computation of the deviation of orbits described by (5)—i.e., that if at least one of the eigenvalues of the matrix

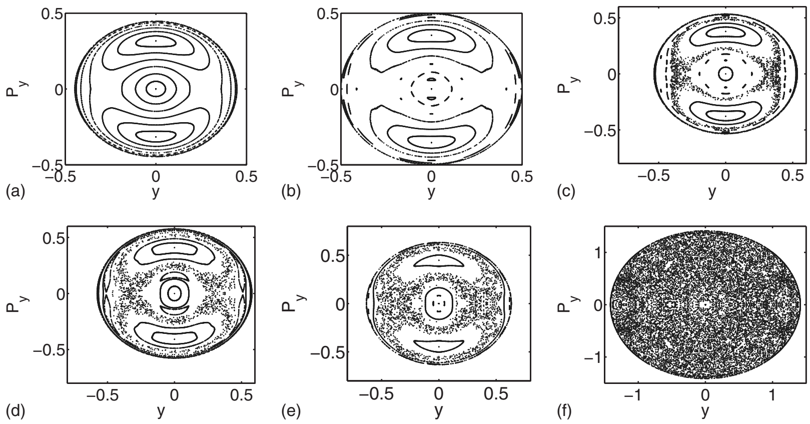

FIG. 1. a), (b) A Poincaré plot in the (y,p_y) plane for $E=\frac{1}{8},\frac{1}{10}$, indicating regular motion. (c), (d), (e) Poincaré plot in the (y,p_y) plane for $E=\frac{1}{7},\frac{1}{6},\frac{1}{5}$, indicating chaotic behavior. (f) A Poincaré plot in the (y,p_y) plane for $E=1$, indicating a strongly chaotic behavior.

$$\mathcal{V}_{\ell i}=\left\{\frac{3}{M^2v^2}\frac{\partial V}{\partial x^\ell}\frac{\partial V}{\partial x^i}+\frac{1}{M}\frac{\partial^2 V}{\partial x^\ell \partial x^i}\right\} \tag{6}$$

is negative, the motion will generally be unstable [this result is not the same as for the computation of the deviation of the orbits generated by (5); it includes information from the curvature of the manifold in which the Hamiltonian motion is embedded].

III. RESULTS AND DISCUSSION

In order to test our theory, we compare our results with the computation regarding the surface of section for a family of models obtained by perturbation of an oscillator Hamiltonian by means of higher-order polynomials.

One may easily verify that the unperturbed two-dimensional oscillator potential is predicted to be stable.

We take for illustration here a simple and important case of coupled harmonic oscillators with perturbation for which the potential is given by

$$V(x,y)=\frac{1}{2}(x^2+y^2)+6x^2y^2. \tag{7}$$

Figure 1 shows the surface of section for several energies. Figures 1(a) and 1(b), corresponding to $E=\frac{1}{10},\frac{1}{8}$, show stable orbits, while as the energy increases, as in Figs. 1(c)–1(f), corresponding to $E=\frac{1}{7},\frac{1}{6}\frac{1}{5},1$, the orbits have become more and more unstable and show a chaotic signature.

The calculation using geometrical methods as shown in Fig. 2 is parallel to Fig. 1. Figures 2(a) and 2(b), corresponding to $E=\frac{1}{10},\frac{1}{8}$, show completely positive eigenvalues in the physical region. In comparison, at higher energy, Figs.

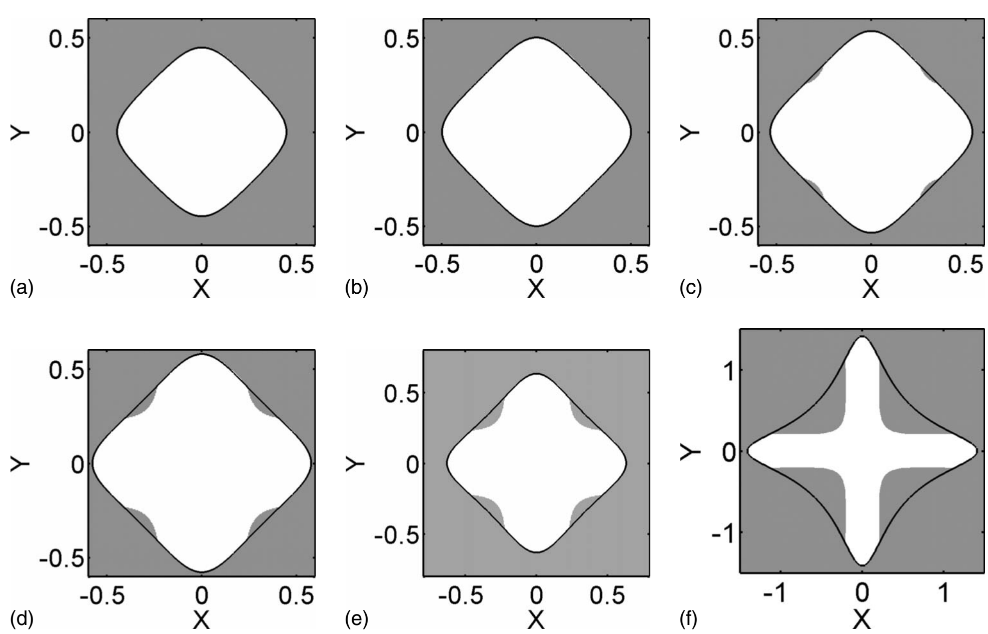

FIG. 2. The dark area shows the region of negative eigenvalues for the matrix $\mathcal{V}$. The light area corresponds to positive eigenvalues where the boundaries are the limits of the physical region. (a), (b) correspond to $E=\frac{1}{8},\frac{1}{10}$. The region of negative eigenvalues does not penetrate the physically accessible region in this case. (c) corresponds to $E=\frac{1}{7}$. The dark areas appear on the boundary of the physically accessible region. (d), (e) correspond to $E=\frac{1}{6},\frac{1}{5}$. The dark areas correspond to the existence of at least one negative eigenvalue in the physical region. (f) The dark area of negative eigenvalues for the matrix $\mathcal{V}$ is seen to penetrate deeply into the light region of physically allowable motions for $E=1$.

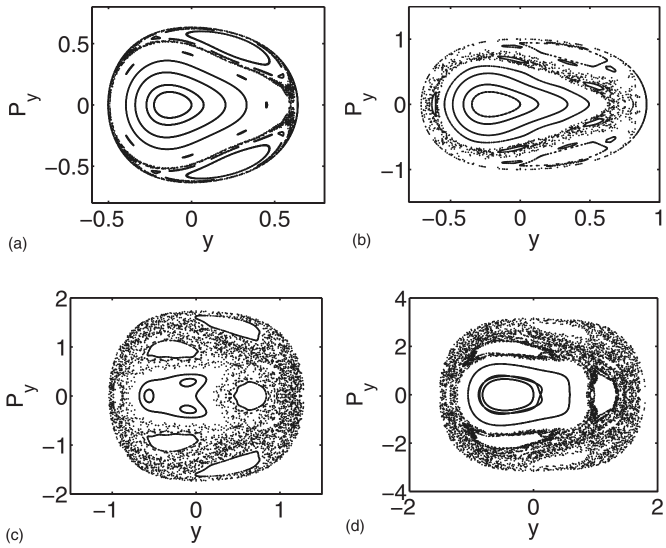

FIG. 3. (a) A Poincaré plot in the (y,p_y) plane for $E=\frac{1}{5}$, indicating regular motion. (b), (c), (d) Poincaré plot in the (y,p_y) plane for $E=\frac{1}{2},\frac{3}{2},5$, indicating chaotic behavior.

2(c)–2(e), corresponding to $E=\frac{1}{7},\frac{1}{6},\frac{1}{5}$, show regions of negative eigenvalues appearing at the boundary of the physical area. These regions increase as the energy increases, and in Fig. 2(f), at $E=1$, the unstable region becomes significant.

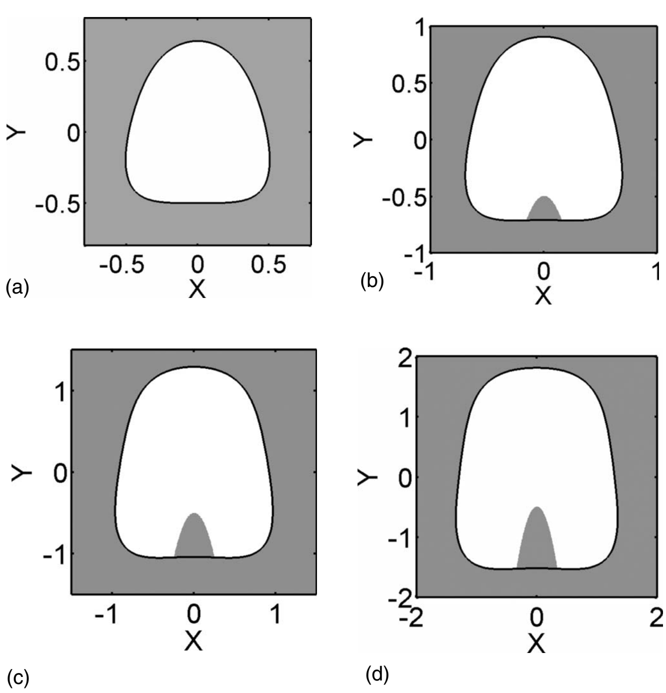

FIG. 4. The dark area shows the region of negative eigenvalues for the matrix $\mathcal{V}$. The lighter area corresponds to positive eigenvalues where the boundaries are the limits of the physical region. (a) corresponds to $E=\frac{1}{5}$. The region of negative eigenvalues will not penetrate the physically accessible region in this case. (b), (c) correspond to $E=\frac{1}{2},\frac{3}{2}$. The dark areas correspond to the existence of at least one negative eigenvalue in the physical region. (d) The dark area of negative eigenvalues for the matrix $\mathcal{V}$ is seen to penetrate deeply into the lighter region of physically allowable motion for $E=5$.

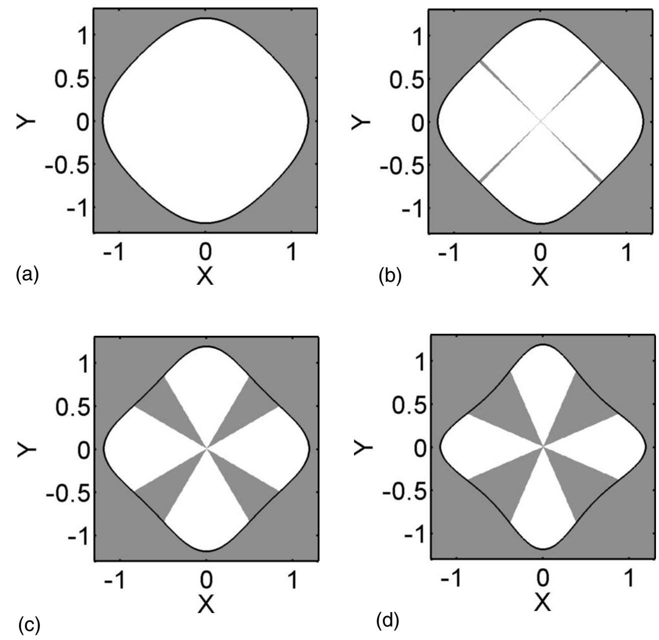

FIG. 5. The dark area shows the region of negative eigenvalues for the matrix $\mathcal{V}$. The lighter area corresponds to positive eigenvalues where the boundaries are the limits of the physical region. (a) corresponds to $\alpha=4$. The region of negative eigenvalues does not penetrate the physically accessible region in this case. (b) corresponds to $\alpha_c=6$. The dark areas appears in the physically accessible region. (c), (d) correspond to $\alpha=8,12$. The dark areas correspond to the existence of at least one negative eigenvalue in the physical region.

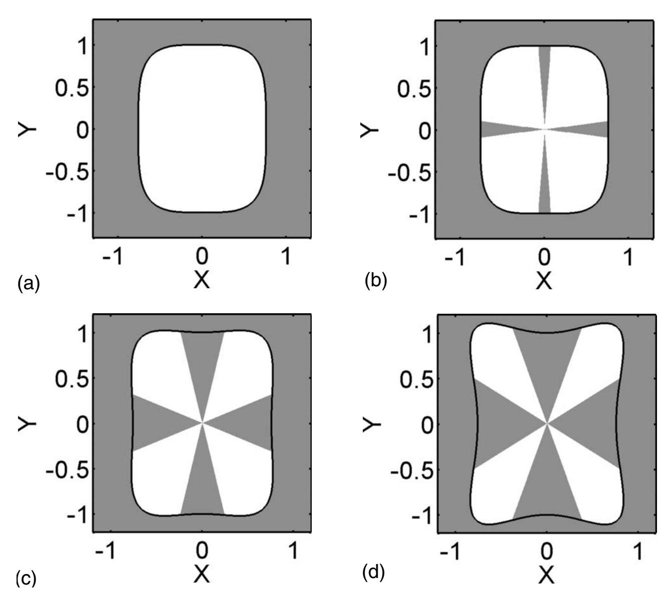

FIG. 6. The dark area shows the region of negative eigenvalues for the matrix $\mathcal{V}$. The lighter area corresponds to positive eigenvalues where the boundaries are the limits of the physical region. (a) corresponds to $\alpha=\frac{1}{10}$. The region of negative eigenvalues does not penetrate the physically accessible region in this case. (b), (c), (d) correspond to $\alpha=-\frac{1}{10},-1,-2$. The dark areas correspond to the existence of at least one negative eigenvalue in the physical region.

Both show $E_c \cong \frac{1}{7}$ as the critical energy of transition.

In addition, we examine our method for a generalization of the Toda potential

$$V(x,y) = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3 + \frac{3}{2}x^4 + \frac{1}{2}y^4. \quad (8)$$

For sufficiently low energy this system shows stabilization, while in the case of higher energies the system becomes less stable and a chaotic signature appears. Figure 3 shows the surface of section for several energies. We can easily see that for $E=\frac{1}{5}$, Fig. 3(a) shows stable orbits; on the other hand, at higher energies $E=\frac{1}{2},\frac{3}{2},5$, Figs. 3(b)–3(d) show clearly unstable regions corresponding to chaotic motion.

In comparison we examine our criterion, Eq. (6), for this potential and find the eigenvalues as a function of space (x,y). Figure 4 shows the result in comparison to Fig. 3. We can see that for $E=\frac{1}{5}$, Fig. 4(a) shows two positive eigenvalues in the physical area, while for energies $E=\frac{1}{2},\frac{3}{2},5$, Figs. 4(b)–4(d) show that there are (one or two) negative eigenvalues in the physical area and hence chaotic motion. Both show $E_c \cong \frac{1}{5}$ as the critical energy of transition.

The transition from order to chaos could depend on a parameter in the potential. The next two examples show how this transition appears in our method. We study the behavior of an important class of quartic oscillators. The first example was treated by a different method by Oloumi and Teychenne [6]. Figure 5 shows the eigenvalues corresponding to

$$V(x,y) = \frac{1}{2}(x^4 + y^4) + \frac{\alpha}{2}x^2 y^2. \quad (9)$$

This system, for $E=1$, shows chaotic behavior for $\alpha > 6$ [6]. The transitions shown in Fig. 5(a) correspond to $\alpha=4$ below the transition, Fig. 5(b) shows the transition at $\alpha_c=4$, and Figs. 5(c) and 5(d), corresponding to $\alpha=8,12$, are above the transition.

Similar behavior can be found for another Hamiltonian of the quartic oscillator:

$$V(x,y) = 3x^4 + y^4 - \alpha x^2 y^2. \quad (10)$$

In this case, the system shows instability for $\alpha < 0$. Figure 6 shows the calculation of the eigenvalues corresponding to different values of α. It clearly shows negative eigenvalues in physical regions above the transition.

IV. CONCLUSION

One can see that in these cases the analytic condition, Eq. (6), gives results in agreement with the numerical technique of surface of section. This condition is exact, sensitive, and easy to apply. In case of two-dimensional Hamiltonian systems, one must only compute eigenvalues of 2×2 symmetric matrices.

The technique of surface of section is also applicable in three dimensions [7], but it is difficult to apply as a criterion for unstable dynamics in this particular case [8]. For such cases one may use the well-known Lyapunov characteristic exponents. BenZion and Horwitz [9] have shown that the geometrical method discussed here used as a much simpler and more efficient criterion for instability.

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